Communications in Algebra®, 43: 43–50, 2015
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ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927872.2014.897188

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ON THE DOT PRODUCT GRAPH OF A COMMUTATIVE RING

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Let A be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and $R = A \times A \times \cdots \times A$ (n times). The total dot product graph of R is the (undirected) graph $TD(R)$ with vertices $R^* = R \setminus \{(0, 0, \ldots, 0)\}$, and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$ (where $x \cdot y$ denote the normal dot product of x and y). Let $Z(R)$ denote the set of all zero-divisors of R. Then the zero-divisor dot product graph of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \setminus \{(0, 0, \ldots, 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. We observe that if $n = 1$, then $TD(R)$ is a disconnected graph and $ZD(R)$ is identical to the well-known zero-divisor graph of R in the sense of Beck–Anderson–Livingston, and hence it is connected. In this paper, we study both graphs $TD(R)$ and $ZD(R)$. For a commutative ring A and $n \geq 3$, we show that $TD(R)$ ($ZD(R)$) is connected with diameter two (at most three) and with girth three. Among other things, for $n \geq 2$, we show that $ZD(R)$ is identical to the zero-divisor graph of R if and only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Key Words: Annihilator graph; Total graph; Zero-divisor graph.

2010 Mathematics Subject Classification: Primary: 13A15; Secondary: 13B99; 05C99.

1. INTRODUCTION

Let R be a commutative ring with nonzero identity, and let $Z(R)$ be its set of zero-divisors. Recently, there has been considerable attention in the literature to associating graphs with algebraic structures (see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [17], [18], [19], [20], [21], [23], [24], [25], and [26]). Probably the most attention has been to the *zero-divisor graph* $\Gamma(R)$ for a commutative ring R. The set of vertices of $\Gamma(R)$ is $Z(R)^* = Z(R) \setminus \{0\}$, and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of a zero-divisor graph goes back to I. Beck [13], who let all elements of R be vertices and was mainly interested in colorings. The zero-divisor graph $\Gamma(R)$ was introduced by David F. Anderson and Philip S. Livingston in [9], where it was shown, among other things, that $\Gamma(R)$ is connected with $\mathrm{diam}(\Gamma(R)) \in \{0, 1, 2, 3\}$ and $\mathrm{gr}(\Gamma(R)) \in \{3, 4, \infty\}$. For a recent survey article on zero-divisor graphs, see [12].

Received May 1, 2013; Revised June 14, 2013. Communicated by E. Houston.
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Let A be a commutative ring with nonzero identity, $1 \leq n < \infty$ be an integer, and let $R = A \times A \times \cdots \times A$ (n times). Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R$. Then the dot product $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in A$. In this paper, we introduce the *total dot product graph* of R to be the (undirected) graph $TD(R)$ with vertices $R^* = R \backslash \{(0, 0, \ldots, 0)\}$, and two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$. Let $Z(R)$ denote the set of all zero-divisors of R. Then the *zero-divisor dot product graph* of R is the induced subgraph $ZD(R)$ of $TD(R)$ with vertices $Z(R)^* = Z(R) \backslash \{(0, 0, \ldots, 0)\}$. It follows that each edge (path) of the classical zero-divisor graph $\Gamma(R)$ is an edge (path) of $ZD(R)$. We observe that if $n = 1$, then $TD(R)$ is a disconnected graph, where $ZD(R)$ is identical to $\Gamma(R)$ in the sense of Beck–Anderson–Livingston, and hence it is connected.

In the second section, for an $1 \leq n < \infty$ and $R = A \times A \times \cdots \times A$ (n times) for some commutative ring A, we show (Theorem 2.2) that $ZD(R) = \Gamma(R)$ if and only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $n = 2$ and A is not an integral domain or $n = 3$ and A is an integral, we show (Theorem 2.3 and Theorem 2.5(1)) that $ZD(R)$ is connected with diameter three. If $n \geq 4$, we show (Theorem 2.5(3)) that $ZD(R)$ is connected with diameter two. If $n \geq 3$, we show (Theorem 2.4) that $TD(R)$ is connected with diameter two. We show (Corollary 2.8)that $ZD(R)$ contains no cycles if and only if $n = 2$ and A is ring-isomorphic to $\mathbb{Z}_2$. We show (Theorem 2.6) that if $n \geq 3$, then the girth of $ZD(R)$ is three (and hence the girth of $TD(R)$ is three).

We recall some definitions. Let Γ be a (undirected) graph. We say that Γ is *connected* if there is a path between any two distinct vertices. For vertices x and y of Γ, we define $d(x, y)$ to be the length of a shortest path from x to y ($d(x, x) = 0$ and $d(x, y) = \infty$ if there is no path). Then the *diameter* of Γ is $diam(\Gamma) = \sup\{d(x, y) \mid x$ and y are vertices of $\Gamma\}$. The *girth* of Γ, denoted by $gr(\Gamma)$, is the length of a shortest cycle in Γ ($gr(\Gamma) = \infty$ if Γ contains no cycles). A graph Γ is *complete* if any two distinct vertices are adjacent.

Throughout, all rings are commutative with nonzero identity. Let R be a commutative ring. Then $Z(R)$ denotes the set of zero-divisors of R, and the distance between two distinct vertices a, b of $TD(R)$ ($ZD(R)$) is denoted by $d_T(a, b)$ ($d_Z(a, b)$). If $ZD(R)$ is identical to $\Gamma(R)$, then we write $ZD(R) = \Gamma(R)$; otherwise, we write $ZD(R) \neq \Gamma(R)$. As usual, $\mathbb{Z}$ and $\mathbb{Z}_n$ will denote the integers and integers modulo n, respectively. Any undefined notation or terminology is standard, as in [22] or [16].

2. BASIC PROPERTIES OF $TD(R)$ AND $ZR(D)$

We start this section with the following result.

Theorem 2.1. *Let A be an integral domain and $R = A \times A$. Then $TD(R)$ is disconnected and $ZD(R) = \Gamma(R)$ is connected. In particular, if A is ring-isomorphic to $\mathbb{Z}_2$, then $ZD(R)$ is complete (i.e., $diam(ZD(R)) = 1$) and $gr(ZD(R)) = \infty$. If A is not ring-isomorphic to $\mathbb{Z}_2$, then $diam(ZD(R)) = 2$ and $gr(ZD(R)) = 4$.*

Proof. Let $B = \{(a, a), (-a, a), (a, -a) \mid a \in A^*\}$, and let $x \in B$. Suppose that $y \in R^*$ and $x \cdot y = 0$. Since A is an integral domain, one can easily see that $y \in B$. Let $M = \{(a, 0), (0, a) \mid a \in A^*\}$ and let $w \in M$. Suppose that $w \cdot s = 0$ for some

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$s \in R^*$. Again, since A is an integral domain, we conclude that $s \in M$. Thus the vertices $(1, 1)$ and $(0, 1)$ are not connected by a path in $TD(R)$. Hence $TD(R)$ is disconnected. Since A is an integral domain, $Z(R)^* = M$. Let $x, y \in M$. Then $x \cdot y = 0$ iff $xy = (0, 0)$. Thus $ZD(R) = \Gamma(R)$. Suppose that A is ring-isomorphic to $\mathbb{Z}_2$. Then it is clear that $diam(ZD(R)) = 1$ and $gr(ZD(R)) = \infty$. Suppose A is not ring-isomorphic to $\mathbb{Z}_2$. Since $ZD(R) = \Gamma(R)$ and A is an integral domain, $diam(ZD(R)) = 2$ by [24, Theorem 2.6] and $gr(ZD(R)) = 4$ by [10, Theorem 2.2]. □

Theorem 2.2. *Let $2 \leq n < \infty$, A be a commutative ring with $1 \neq 0$, and $R = A \times A \times \cdots \times A$ (n times). Then $ZD(R) = \Gamma(R)$ if and only if either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. If $n = 2$ and A is an integral domain, then by Theorem 2.1 we have $ZD(R) = \Gamma(R)$. Suppose that R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then by simple hand-calculations, for every $x, y \in Z(R)^*$, we have $x \cdot y = 0$ iff $xy = (0, 0, 0)$, and hence $ZD(R) = \Gamma(R)$.

Conversely, suppose that $ZD(R) = \Gamma(R)$. Assume that A is not an integral domain. Then there is an $a \in Z(A)^*$. Hence $x = (1, a, 0, 0, \ldots, 0)$, $y = (a, -1, 0, 0, \ldots, 0) \in Z(R)^*$, and $x \cdot y = 0$, but $xy \neq (0, 0, \ldots, 0)$. Thus $x - y$ is an edge of $ZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction. Thus A must be an integral domain. Now assume that $n = 3$ and A is not ring-isomorphic to $\mathbb{Z}_2$. Then there is an $a \in A \backslash \{0, 1\}$. Let $x = (1, a, 0)$ and $y = (-a, 1, 0)$. Then $x \neq y$ and it is clear that $x - y$ is an edge of $ZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction again. Hence assume that $n \geq 4$. Let $x = (1, 1, 0, 1, 0, 0, \ldots, 0)$ and $y = (-1, 1, 1, 0, 0, \ldots, 0)$. Then $x \neq y$, $x \cdot y = 0$, but $xy \neq (0, 0, \ldots, 0)$, a contradiction. Thus we conclude that either $n = 2$ and A is an integral domain or R is ring-isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

In view of Theorem 2.1, we have the following result.

Theorem 2.3. *Let A be a commutative ring with $1 \neq 0$ that is not an integral domain, and let $R = A \times A$. Then the following statements hold:*

(1) *$TD(R)$ is connected and $diam(TD(R)) = 3$;*
(2) *$ZD(R)$ is connected, $ZD(R) \neq \Gamma(R)$, and $diam(ZD(R)) = 3$;*
(3) *$gr(ZD(R)) = gr(TD(R)) = 3$.*

Proof. **(1).** Let $x = (a, b)$, $y = (c, d) \in R^*$, where $x \neq y$, and assume that $x \cdot y \neq 0$. Since A is not an integral domain, there are $f, g \in A^*$ (not necessarily distinct) such that $fg = 0$. Let $w = (-bf, af)$ and $v = (-dg, cg)$. Note that $w, v \in Z(R)$. Clearly $x \cdot w = w \cdot v = v \cdot y = 0$. Since $x \cdot y \neq 0$, $w \neq y$ and $v \neq x$. First, assume that $v, w \in Z(R)^*$. If $x \cdot y = 0$ or $y.w = 0$, then $x - v - y$ or $x - w - y$ is a path of length 2 in $TD(R)$ from x to y. Assume that neither $x \cdot y = 0$ nor $y \cdot w = 0$. Then x, w, v, y are distinct, and hence $x - w - v - y$ is a path of length 3 in $TD(R)$ from x to y. Now assume that $w = (0, 0)$ or $v = (0, 0)$. If $w = (0, 0)$, then replace w by $(f, -f) \in Z(R)^*$, and hence $x \cdot w = (a, b) \cdot (f, -f) = 0$. Similarly, if $v = (0, 0)$, then replace v by $(g, -g) \in Z(R)^*$. Hence if $w = (0, 0)$ or $v = (0, 0)$, then we are able to redefine w and v so that $w, v \in Z(R)^*$ and $x \cdot w = w \cdot v = v \cdot y = 0$. Thus as in the earlier argument, we conclude that there is a path of length at most 3 in $TD(R)$ from x to y. Thus $TD(R)$ is connected and $d_T(x, y) \leq 3$ for every $x, y \in R^*$. Now, let $x = (1, 1)$ and $y = (1, 0)$. We show $d_T(x, y) = 3$, and hence $diam(TD(R)) = 3$. Let $w \in R^*$ such that $x \cdot w = 0$. Then $w = (a, -a)$ for some $a \in A^*$. Since $w \cdot y = a \neq 0$, $d_T(x, y) > 2$. Hence $d_T(x, y) = 3$. In particular, let $k, t \in A^*$ such that $kt = 0$, $w = (k, -k)$, and $v = (0, t)$. Then $x - w - v - y$ is a path of length 3 in $TD(R)$ from x to y.

(2). Since A is not an integral domain, $ZD(R) \neq \Gamma(R)$ by Theorem 2.2. Let $x, y \in Z(R)^*$, and assume that $x \cdot y \neq 0$. In view of the proof of (1), we are able to find $w, v \in Z(R)^*$ such that either $x - w - y$ is a path in $ZD(R)$ or $x - v - y$ is a path in $ZD(R)$ or $x - w - v - y$ is a path in $ZD(R)$. Hence $diam(ZD(R)) \leq 3$. Let $a \in Z(A)^*$. Then $x = (1, a), y = (0, 1) \in Z(R)^*$. We show $d_Z(x, y) = 3$, and thus $diam(ZD(R)) = 3$. Since $x \cdot y \neq 0$, $d_Z(x, y) > 1$. Suppose there is a $v = (g, h) \in Z(R)^*$ such that $x - v - y$ is a path of length 2 in $ZD(R)$ from x to y. Since $v \cdot y = 0$, we have $h = 0$, and hence $v = (g, 0)$. Since $x \cdot y = 0$, we have $g = 0$, and thus $v = (0, 0)$, a contradiction. Thus $d_Z(x, y) = 3$, and hence $diam(ZD(R)) = 3$.

(3). Since A is not an integral domain, there are $a, b \in A^*$ (not necessarily distinct) such that $ab = 0$. Then $x = (a, 0), y = (0, b), w = (b, a) \in Z(R)^*$. Hence $x - y - w - x$ is a cycle of length 3 in $ZD(R)$. Thus $gr(TD(R)) = gr(ZD(R)) = 3$. □

Theorem 2.4. *Let A be a commutative ring with* $1 \neq 0$, $3 \leq n < \infty$, *and let* $R = A \times A \times \cdots \times A$ (*n times*). *Then* $TD(R)$ *is connected and* $diam(TD(R)) = 2$.

Proof. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in R^*$, and suppose that $x \cdot y \neq 0$. Then let $M = \{i \mid x_i = y_i = 0, 1 \leq i \leq n\}$. Suppose that M is not the empty set. Then choose a $k \in M$, and let $w = (w_1, \ldots, w_n) \in R^*$, where $w_k = 1$ and $w_i = 0$ if $i \neq k$. Then $x - w - y$ is a path of length 2 in $TD(R)$ from x to y. Thus suppose that M is the empty set. Then let $f(x) = min\{i \mid x_i \neq 0, 1 \leq i \leq n\}$ and $f(y) = min\{i \mid y_i \neq 0, 1 \leq i \leq n\}$. Since M is the empty set, we conclude that $f(x) = 1$ or $f(y) = 1$. We may assume that $f(x) = 1$. Let $v = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1, 0, 0, 0, \ldots, 0) \in R$. Suppose that $v \neq (0, \ldots, 0)$. Then it is easy to check that $x \cdot y = v \cdot y = 0$. Since $x \cdot y \neq 0$, $v \neq x$ and $v \neq y$. Hence $x - v - y$ is a path of length 2 in $TD(R)$ from x to y. Suppose that $v = (0, \ldots, 0)$. Then $x_1y_2 - x_2y_1 = 0$. Let $w = (-x_2,\ x_1, 0, 0, \ldots, 0) \in R$. Since $x_1 \neq 0$, $w \in R^*$. Hence $x \cdot w = -x_1x_2 + x_1x_2 = 0$ and $w \cdot y = x_1y_2 - x_2y_1 = 0$. Since $x \cdot w = w \cdot y = 0$ and $x \cdot y \neq 0$, $x \neq w$ and $y \neq w$. Thus $x - w - y$ is a path of length 2 in $TD(R)$ from x to y. Hence $TD(R)$ is connected and $diam(TD(R)) = 2$. □

Theorem 2.5. *Let A be a commutative ring with* $1 \neq 0$. *Then the following statements hold*:

(1) *If A is an integral domain and* $R = A \times A \times A$, *then* $ZD(R)$ *is connected* ($ZD(R) \neq \Gamma(R)$ *by Theorem* 2.2) *and* $diam(ZD(R)) = 3$;
(2) *If A is not an integral domain and* $R = A \times A \times A$, *then* $ZD(R)$ *is connected* ($ZD(R) \neq \Gamma(R)$ *by Theorem* 2.2) *and* $diam(ZD(R)) = 2$;
(3) *If* $4 \leq n < \infty$ *and* $R = A \times A \times \cdots \times A$ (*n times*), *then* $ZD(R)$ *is connected* ($ZD(R) \neq \Gamma(R)$ *by Theorem* 2.2) *and* $diam(ZD(R)) = 2$.

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Proof. **(1).** Since $\Gamma(R)$ is connected and every path in $\Gamma(R)$ is a path in $ZD(R)$, we conclude that $ZD(R)$ is connected. Since $diam(ZD(R)) \leq diam(\Gamma(R))$ and $diam(\Gamma(R)) = 3$ by [24, Theorem 2.6], we conclude that $diam(ZD(R)) \leq 3$. Let $x = (1, 0, -1), y = (0, 1, -1) \in Z(R)^*$. Then $x \cdot y = 1 \neq 0$. We show $d_Z(x, y) = 3$. Let $w = (w_1, w_2, w_3) \in R$ such that $x \cdot w = w \cdot y = 0$. Then a trivial calculation leads to $w_1 = w_2 = w_3$. Since A is an integral domain, $w \in Z(R)$ if and only if $w = (0, 0, 0)$. Thus $d_z(x, y) = 3$. Hence $diam(ZD(R)) = 3$.

(2). (Similar to the proof of Theorem 2.4). Let $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in Z(R)^*$, and suppose that $x \cdot y \neq 0$. Then let $M = \{i \mid x_i = y_i = 0, 1 \leq i \leq 3\}$. Suppose that M is not the empty set. Then choose a $k \in M$, and let $w = (w_1, w_2, w_3) \in Z(R)^*$, where $w_k = 1$ and $w_i = 0$. If $i \neq k$, then $x - w - y$ is a path of length 2 in $ZD(R)$ from x to y. Thus suppose that M is the empty set. Then let $f(x) = min\{i \mid x_i \neq 0, 1 \leq i \leq n\}$ and $f(y) = min\{i \mid y_i \neq 0, 1 \leq i \leq 3\}$. Since M is the empty set, we conclude that $f(x) = 1$ or $f(y) = 1$. We may assume that $f(x) = 1$. Let $v = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1) \in R$. Suppose that $v \in Z(R)^*$. Then it is easy to check that $x \cdot y = v \cdot y = 0$. Since $x \cdot y \neq 0$, $v \neq x$ and $v \neq y$. Hence $x - v - y$ is a path of length 2 in $ZD(R)$ from x to y. Suppose that $v \notin Z(R)$. Then choose an $a \in Z(A)^*$. Then $av \in Z(R)^*$ and $x - av - y$ is a path of length 2 in $ZD(R)$ from x to y. Suppose that $v = (0, 0, 0)$. Then $x_1y_2 - x_2y_1 = 0$. Let $w = (-x_2, x_1, 0) \in Z(R)$. Since $x_1 \neq 0$, $w \in Z(R)^*$. Hence $x \cdot w = -x_1x_2 + x_1x_2 = 0$ and $w \cdot y = x_1y_2 - x_2y_1 = 0$. Since $x \cdot w = w \cdot y = 0$ and $x \cdot y \neq 0$, $x \neq w$ and $y \neq w$. Thus $x - w - y$ is a path of length 2 in $ZD(R)$ from x to y. Hence $ZD(R)$ is connected and $diam(ZD(R)) = 2$.

(3). The proof is similar to the proof of Theorem 2.4. Just observe that if $n \geq 4$, then v as in the proof of Theorem 2.4 is in $Z(R)$. □

Theorem 2.6. *Let A be a commutative ring with $1 \neq 0$, $3 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then $gr(ZD(R)) = gr(TD(R)) = 3$.*

Proof. Let $a = (1, 0, \ldots, 0), b = (0, 1, 0, \ldots, 0)$, and $c = (0, 0, 1, 0, \ldots, 0)$. Then $a - b - c - a$ is a cycle of length 3. □

Corollary 2.7. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $gr(ZD(R)) = 3$;
(2) $gr(TD(R)) = 3$;
(3) *A is not an integral domain and $n = 2$ or $n \geq 3$.*

Proof. This is clear by Theorem 2.3 and Theorem 2.6. □

Corollary 2.8. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $gr(ZD(R)) = \infty$;
(2) *A is ring-isomorphic to $\mathbb{Z}_2$ and $n = 2$;*
(3) $diam(ZD(R)) = 1$.

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Proof. (1) ⇒ (2). Suppose $gr(ZD(R)) = \infty$. Then $n = 2$ by Theorem 2.6. Hence A is an integral domain by Corollary 2.7. Hence $ZD(R) = \Gamma(R)$ by Theorem 2.2. Thus A is ring-isomorphic to $\mathbb{Z}_2$ by [10, Theorem 2.4]. (2) ⇒ (3). It is clear. (3) ⇒ (1). Since $diam(ZD(R)) = 1$, we conclude that $n = 2$ and A is an integral domain by Theorems 2.3 and 2.5. Thus A is ring-isomorphic to $\mathbb{Z}_2$ by Theorem 2.1. Thus $gr(ZD(R)) = \infty$. □

Corollary 2.9. *Let A be a commutative ring with $1 \neq 0$ such that A is not ring-isomorphic to $\mathbb{Z}_2$, $0 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $gr(ZD(R)) = 4$;
(2) $ZD(R) = \Gamma(R)$;
(3) *$TD(R)$ is disconnected*;
(4) *$n = 2$ and A is an integral domain.*

Proof. This is clear by Theorem 2.1, Theorem 2.2, Corollary 2.7, and Corollary 2.8. □

Corollary 2.10. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $diam(ZD(R)) = 3$;
(2) *Either A is not an integral domain and $n = 2$ or A is an integral domain and $n = 3$.*

Proof. This is clear by Theorem 2.1, Theorem 2.3, and Theorem 2.5. □

Corollary 2.11. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $diam(ZD(R)) = 2$;
(2) *Either A is an integral domain that is not ring-isomorphic to $\mathbb{Z}_2$ and $n = 2$, A is not an integral domain, and $n = 3$, or $n \geq 4$.*

Proof. This is clear by Theorem 2.1, Theorem 2.5, and Corollary 2.10. □

Corollary 2.12. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then $diam(TD(R)) = 3$ if and only if A is not an integral domain and $n = 2$.*

Proof. This is clear by Theorem 2.1, Theorem 2.3, and Theorem 2.4. □

Corollary 2.13. *Let A be a commutative ring with $1 \neq 0$, $2 \leq n < \infty$, and $R = A \times A \times \cdots \times A$ (n times). Then the following statements are equivalent:*

(1) $diam(TD(R)) = 2$;
(2) *$TD(R)$ is connected and $n \geq 3$*;
(3) $n \geq 3$.

Proof. The proof is clear by Theorem 2.3 and Theorem 2.4. □

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Corollary 2.14. *Let A be a commutative ring with* $1 \neq 0$, $2 \leq n < \infty$, *and* $R = A \times A \times \cdots \times A$ (*n times*). *Then* $diam(TD(R)) = diam(ZD(R)) = 3$ *if and only if A is not an integral domain and* $n = 2$.

Proof. This is clear by Corollary 2.10 and Corollary 2.12. □

ACKNOWLEDGMENT

I would like to thank the referee for several helpful suggestions.

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REFERENCES

[1] Akbari, S., Maimani, H. R., Yassemi, S. (2003). When a zero-divisor graph is planar or a complete r-partite graph. *J. Algebra* 270:169–180.

[2] Alizadeh, M., Dasb, A. K., Maimanic, H. R., Pournaki, M. R., Yassemi, S. (2012). On the diameter and girth of zero-divisor graphs of posets. *Disc. Appl. Math.* 160: 1319–1324.

[3] Anderson, D. F. (2008). On the diameter and girth of a zero-divisor graph, II. *Houston J. Math.* 34:361–371.

[4] Anderson, D. F., Badawi, A. (2008). On the zero-divisor graph of a ring. *Comm. Algebra* (36):3073–3092.

[5] Anderson, D. F., Badawi, A. (2008). The total graph of a commutative ring. *J. Algebra* 320:2706–2719.

[6] Anderson, D. F., Badawi, A. (2012). The total graph of a commutative ring without the zero element. *J. Algebra Appl.* 11(4):2500740. (18pgs). DOI:10.1142/S0219498812500740.

[7] Anderson, D. F., Badawi, A. (2013). The generalized total graph of a commutative ring. *J. Algebra Appl.* 12(5):1250212. (18pgs). DOI:10.1142/S021949881250212X.

[8] Anderson, D. F., LaGrange, J. D. (2012). Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph. *J. Pure Appl. Algebra* 216:1626–1636.

[9] Anderson, D. F., Livingston, P. S. (1999). The zero-divisor graph of a commutative ring. *J. Algebra* 217:434–447.

[10] Anderson, D. F., Mulay, S. B. (2007). On the diameter and girth of a zero-divisor graph. *J. Pure Appl. Algebra* 210:543–550.

[11] Anderson, D. F., Levy, R., Shapiro, J. (2003). Zero-divisor graphs, von Neumann regular rings, and Boolean algebras. *J. Pure Appl. Algebra* 180:221–241.

[12] Anderson, D. F., Axtell, M. C., Stickles, J. A. Jr., (2011). Zero-divisor graphs in commutative rings. In: Fontana, M., Kabbaj, S.-E., Olberding, B., Swanson, I., eds. *Commutative Algebra, Noetherian and Non-Noetherian Perspectives*. New York: Spring-Verlag, pp. 23–45.

[13] Beck, I. (1988). Coloring of commutative rings. *J. Algebra* 116:208–226.

[14] Behboodi, M., Rakeei, Z. (2011). The annihilator-ideal graph of commutative rings I. *J. Algebra Appl.* 10(4):727–739.

[15] Behboodi, M., Rakeei, Z. (2011). The annihilator-ideal graph of commutative rings II. *J. Algebra Appl.* 10(4):741–753.

[16] Bollaboás, B. (1998). *Modern Graph Theory*. New York: Spring-Verlag.

[17] Ghalandarzadeh, Shirinkam, S., Rad, P. M. (2013). Annihilator ideal-based zero-divisor graphs over multiplication modules. *Comm. Algebra* 41(3):1134–1148.

[18] Tamizh Chelvam, T., Asir, T. (2011). Domination in the total graph on Z_n. *Discrete Math. Algorithms Appl.* 3(4):413–421.

[19] Tamizh Chelvam, T., Asir, T. (2013). The intersection graph of gamma sets in the total graph of a commutative ring I. *J. Algebra Appl.* 12(4):1250198. (18pgs). DOI:10.1142/S0219498812501988.

[20] Tamizh Chelvam, T., Asir, T. (2013). The intersection graph of gamma sets in the total graph of a commutative ring II. *J. Algebra Appl.* 12(4):1250199. (14pgs). DOI:10.1142/S021949881250199X.

[21] Chiang-Hsieh, H.-J. (2008). Classification of rings with projective zero-divisor graphs. *J. Algebra* 319:2789–2802.

[22] Huckaba, J. A. (1988). *Commutative Rings with Zero Divisors*. New York/Basel: Marcel Dekker.

[23] Maimani, H. R., Pournaki, M. R., Tehranian, A., Yassemi, S. (2011). Graphs attached to rings revisited. *Arab. J. Sci. Eng.* 36:997–1012.

[24] Lucas, T. G. (2006). The diameter of a zero-divisor graph. *J. Algebra* 301:3533–3558.

[25] Wang, H.-J. (2006). Zero-divisor graphs of genus one. *J. Algebra* 304:666–678.

[26] Wickham, C. (2009). Rings whose zero-divisor graphs have positive genus. *J. Algebra* 321:377–383.

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